Dynamics of a system of sticking particles of a finite size on the line

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Abstract

The continuous limit of large systems of particles of finite size on the line is described. The particles are assumed to move freely and stick under collision, to form compound particles whose mass and size is the sum of the masses and sizes of the particles before collision, and whose velocity is determined by conservation of linear momentum.

1 Introduction

1.1 Review

Models of point particles on the line which stick under collision have recently been considered in the literature, starting from the pioneering paper of Zeldovich from 1970 ([11], see also [6]). This model is extremely simple: Imagine a swarm of point particles moving without interaction (constant velocity) on the line. When two (or more) particles collide, they stick together and continue to move at a constant velocity, determined by conservation of their initial momentum.

Assuming for simplicity there are initially N identical particles of mass $1/N$, we can describe the density and momentum of this swarm by the fields

$$\rho_N = N^{-1}\sum_1^N \delta_{(x-x_i(t))}, \quad \rho_N u_N = N^{-1}\sum_1^N v_i(t)\delta_{(x-x_i(t))} \ , \tag{1.1}$$

where here δ stands for the Dirak delta-function. Here $x_i(t)$ is the position of the i-th particle at time t and $v_i(t)$ is its velocity at that instance, assumed to be constant between collisions. The assumption of sticking collisions can be stated as

$$v_i(t+) = \frac{\sum_{j;\ x_j(t)=x_i(t)} v_j(t)}{\#\{j;\ x_j(t)=x_i(t)\}} \ , x_i(t) = x_i(0) + \int_0^t v_i(s)ds \ . \tag{1.2}$$

Let (ρ_N, u_N), $N \to \infty$ be a sequence of the form (1.1). It is said to converge weakly to (ρ, u) if

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \rho_N \phi dx = \int_{-\infty}^{\infty} \rho\phi dx \ , \ \lim_{N\to\infty} \int_{-\infty}^{\infty} \rho_N u_N \phi dx = \int_{-\infty}^{\infty} \rho u \phi dx \tag{1.3}$$

for any $\phi \in C_0(R)$ and $t \geq 0$. The first rigorous treatment of this limit was considered in [4]. It was proved that a sequence (ρ_N, u_N) of point particles (of the form (1.1)) converges weakly to a weak solution (ρ, u) of the *zero pressure gas dynamics* system

$$\frac{\partial \rho}{\partial t} + \frac{\partial(u\rho)}{\partial x} = 0 \quad ; \quad \frac{\partial(\rho u)}{\partial t} + \frac{(\partial \rho u^2)}{\partial x} = 0 \ , (x,t) \in (-\infty, \infty) \times (0, \infty) \tag{1.4}$$

provided (1.3) is satisfied for $t = 0$ (and some additional, technical, conditions).

Apart from being a continuous limit of the model of point particles, the zero-pressure gas dynamics (1.4) attracted a considerable interest by its own. It is an example of hyperbolic system of a pair of conservation laws which is degenerate, in the sense that the two systems of characteristics coincide (see [8]). This degeneracy leads to a special type of singular solutions, called $\delta-$shocks, studied by several authors.

The $\delta-$shock solutions present a challenge for the study of (1.4), and motivate the study of measure valued solutions of this system, their existence and uniqueness (see, e.g. [1], [5] [9], [7] and ref. therein). Unlike the non-degenerate gas dynamics systems, the entropy condition is not enough to guarantee uniqueness of the weak solutions for (1.4). However, the evolution of a finite number of sticking particles is evidently determined uniquely by the initial conditions $x_i(0), v_i(0)$. As established in [4], the solution of (1.4) is unique, as a weak limit $N \to \infty$ of the point particles dynamics (1.1), and depends only on the weak limit of the initial data $\rho(x, 0), u(x, 0)$ (and not on the particular sequence).

It is evident that such a result cannot be extended to higher space dimension. Indeed, the collision of a pair of point particles in the space of dimension $d > 1$ is a non-generic event. This leads to an apparent paradox. The sticking particle dynamics, which is a very natural model, cannot converge into a deterministic macroscopic process in the limit of large particle numbers, unless the space dimension is one. A way to circumvent such a paradox and obtain, perhaps, a macroscopic limit in higher dimension is to replace the assumption of point particles by the assumptions that the particles posses a finite size, scaled appropriately with respect to N.

In this paper we attempt to consider the macroscopic limit of a swarm of particles of finite size. However, we still restrict ourselves to particles on the line. We show that a macroscopic limit exists and is unique in this case, even though such a limit cannot be described by the zero pressure system (1.4). We also describe this macroscopic limit explicitly. The extension of this model to higher dimension is a challenge we hope to meet sometime in the future.

In the rest of this section (section 1.2) we describe the setting of the problem for swarm of N particles of finite size and mass $1/N$, and formulate the main result. Unlike the case of point particles, there is no explicit Eulerian description of the limit $N \to \infty$, as the zero pressure gas dynamics (1.4) for the system of point particles. To formulate the limit explicitly we need a Lagrangian description of this system. Such a description was introduced in [2] for point particles, and takes the form of a scalar conservation law for the mass cumulation function.

This representation is reviewed in part 2.2 of Section 2 below. In part 2.1 we introduce our main result in an explicit way (Theorem 1), taking advantage of the Lagrangian description.

The proof of the main result involves some extensions of elementary results and well known definitions from convex analysis. For the convenience of the reader we collected these definitions and results in section 3. The proof of the main result is given in section 4. The proof of the auxiliary results of section 3 is given in section 5.

1.2 Point particles of finite size

Consider N identical particles of fixed size ν and mass density ε^{-1} on the line. The mass of any each particle is ν/ε. We shall assume a total unit mass, so $N\nu/\varepsilon = 1$. The density profile of such a particle whose center is at the origin is given by

$$h_\nu(x) = \begin{cases} \varepsilon^{-1} & |x| \leq \nu/2 \\ 0 & |x| > \nu/2 \end{cases}$$

The mass distribution of the system at time t is described by the density

$$\rho_\nu(x,t) = \sum_1^N h_\nu(x - x_i(t)) \ , \tag{1.5}$$

where $x_{i+1}(t) \geq x_i(t) + \nu$, $1 \leq i \leq N-1$ are the positions of the particles at time t. The velocity field is

$$u_\nu(x,t) = \varepsilon \sum_1^N v_i(t) h_\nu(x - x_i(t)) \tag{1.6}$$

where $v_i(t)$ is the velocity of the *ith* particle. Particles are assumed to move at constant velocity, as long as they do not collide. If a pair of particles collides then they stick together to form a compound particle whose mass and size is the sum of the corresponding masses and sizes of the particles before collision. The velocity of the compound particle after collision is determined by the conservation of linear momentum, and is constant in time between collisions. This law can be described as

$$v_i(t+) = \frac{\sum_j 1_\nu\left(x_i(t) - x_j(t); |i-j|\right) v_j(t)}{\sum_j 1_\nu\left(x_i(t) - x_j(t); |i-j|\right)} \ ,$$

$$x_i(t) = x_i(0) + \int_0^t v_i(s) ds \ . \tag{1.7}$$

where $1_\nu(x, y; j) = 1$ if $|x - y| = j\nu$, $1_\nu(x, y; j) = 0$ otherwise.

Remark 1.1. *Note that* (1.7) *implies that the order of the particles on the line is preserved. Moreover, if the collision time between particles $i, i+1$ is t_0, then these particles are glued to each other, and move under the same velocity, for any $t > t_0$.*

The object of this paper is to extend the convergence result of (1.1) to a system of particles (1.5, 1.6) of finite, shrinking size $\nu_N \searrow 0$. We shall prove

Main Result: *The sequence* (ρ_N, u_N) *given by (1.5, 1.6) where* $\nu = \varepsilon/N$ *converges weakly, under some additional assumptions (see Theorem 1, section 2.1), to a pair of functions* (ρ, u) *provided* (1.3) *is satisfied at* $t = 0$. *Moreover,* (ρ, u) *depends only on* $\rho(\cdot, 0), u(\cdot, 0)$.

2 Main result

2.1 Explicit formulation of the main result

Here we formulate the explicit form of the limit claimed at the end of Section 1.2. Before this we need some new definitions:

Definition 2.1. *A function* f *is said to be* $\varepsilon-$*convex on the interval* $[0,1]$ *if* $m \to f(m) - \varepsilon m^2/2$ *is convex on this interval. The set of all* $\varepsilon-$ *convex functions on* $[0,1]$ *is called* $CON_\varepsilon[0,1]$.

Definition 2.2. *The* $\varepsilon-$*convex hull of a function* f *on* $[0,1]$ *is*

$$f_\varepsilon(m) := \sup_{\phi \in CON_\varepsilon[0,1]} \{\phi(m) \ ; \ \phi \le f\}$$

Recall the definition of the Legendre Transform:

Definition 2.3. *Let* $\Phi : \mathbb{R} \to \mathbb{R}$. $\Phi^* : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ *is the Legendre Transform of* Φ *given by*

$$\Phi^*(m) := \sup_{x \in \mathbb{R}} \{xm - \Phi(x)\} \ .$$

Theorem 1. *Let a sequence* (ρ_N, u_N) *given by (1.5, 1.6) where* $\nu = \varepsilon/N$. *Let* $(\overline{\rho}_N, \overline{u}_N) := (\rho_N(\cdot, 0), u_N(\cdot, 0))$, $(\overline{\rho}, \overline{u})$ *the weak limit of* $(\overline{\rho}_N, \overline{u}_N)$. *Assume*

$$supp(\overline{\rho}_N) \subset K \quad ; \quad \|\overline{u}_N\|_\infty < C \quad N = 1, 2, \dots \ , \tag{2.1}$$

holds for some compact $K \subset \mathbb{R}$ *and* $C > 0$, *for any* $N = 1, 2, \dots$. *Assume, in addition*

$$\|\overline{\rho}\|_\infty < \varepsilon^{-1} \ .$$

Set

$$\overline{M}(x) = \int_{-\infty}^x \overline{\rho}(s) ds \ ; \ \overline{\Phi}(x) = \int_{-\infty}^x \overline{M}(s) ds \quad ; \quad \overline{\Psi} = \overline{\Phi}^* \ ,$$

$$v(m) = \overline{u}\left(\partial_m \overline{\Psi}\right) \ ; \ \ \overline{V}(m) = \int_0^m v(s) ds \ ,$$

$$\Psi(m, t) = \left(\overline{\Psi} + t\overline{V}\right)_\varepsilon (m) \ , \ \Phi(, t) := \Psi^*(, t) \ . \tag{2.2}$$

Then

$$\rho(x, t) = \partial_x^2 \Phi(x, t) \quad ; \quad u(x, t) = v\left(\partial_x \Phi(x, t)\right) \ . \tag{2.3}$$

is the weak limit of (ρ_N, u_N).

2.2 Lagrangian description

In [2] the weak solution of (1.4) was interpreted in terms of Lagrange coordinates. Let $M = M(x,t)$ be an entropy solution of the scalar conservation law.

$$\frac{\partial M}{\partial t} + \frac{\partial \overline{V}(M)}{\partial x} = 0 \quad , \quad M(x,0) = \overline{M}(x). \tag{2.4}$$

Equation (2.4) can be integrated once. Setting

$$\Phi(x,t) = \int_{-\infty}^{x} M(s,t)ds, \tag{2.5}$$

(2.4) takes the form of the Hamilton-Jacobi equation

$$\frac{\partial \Phi}{\partial t} + \overline{V}\left(\frac{\partial \Phi}{\partial x}\right) = 0 \quad ; \quad \Phi(x,0) = \overline{\Phi}(x) := \int_{-\infty}^{x} \overline{M}_(s)ds \tag{2.6}$$

The *viscosity solution* (see, e.g. [3]) of (2.6) is given by

$$\Phi(x,t) = \left(\overline{\Psi} + t\overline{V}\right)^{*}(x) \tag{2.7}$$

where $\overline{\Psi} = \overline{\Phi}^{*}$. Since Φ is a convex function of x for any t by definition, $\partial_x^2\Phi$ exists as a measure on $\mathbb{R}$ for any $t > 0$. Taking the Legendre transform of Ψ we obtain

$$\Psi(m,t) = \left(\overline{\Psi} + t\overline{V}\right)^{**}(m) \tag{2.8}$$

where $\Psi = \Phi^{*}$ for any fixed t.

It is shown in [2] that the weak solution of (1.4) satisfies

$$\rho(x,t)dx = \partial_x^2\Phi(x,t) \quad ; \quad u(x,t) = \overline{V}^{'}\left(\partial_x\Phi(x.t)\right) \ .$$

Remark 2.1. *Recall that for any function $f : [0,1] \to \mathbb{R}$, the convex hull f_0 is defined by*

$$f_0(m) := \sup_{\phi \in CON_0[0,1]} \{\phi(m) \ ; \ \phi \le f\}$$

where $CON_0([0,1]$ is the set of all convex functions on $[0,1]$ (consistent with Definition 2.1) . The convex hull of a function is obtained by applying twice the Legendre transform: $f^{**} = f_0$, so (2.8) can be written as

$$\Psi(m,t) = \left(\overline{\Psi} + t\overline{V}\right)_0(m) \ . \tag{2.9}$$

Compare (2.9) to (2.2).

3 Auxiliary results and definitions

Recall Definitions 2.1,2.2.

Lemma 3.1. *The following conditions are equivalent*

i) $f \in CON_\varepsilon[0,1]$.

ii) for any $m_1 < m_2$ *in* $[0,1]$ *and any* $s \in [0,1]$,

$$f\left(sm_1 + (1-s)m_2\right) \geq sf(m_1) + (1-s)f(m_2) + \varepsilon s(s-1)/2 \ .$$

iii) $f = f_\varepsilon$.

Definition 3.1. *An* $\varepsilon-$*parabola is the graph of a quadratic function* $P = P(s)$ *on* $[0,1]$ *where* $d^2P/ds^2 \equiv \varepsilon$.

Remark 3.1. *Lemma 3.1-(ii) can be stated as follows:* $f \in CON_\varepsilon[0,1]$ *if and only if the graph of* f *on any interval* $(m_1, m_2) \subset [0,1]$ *is below any* $\varepsilon-$*parabola on the same interval which connect the points* $(m_1, f(m_1))$ *and* $(m_2, f(m_2))$.

From this remark we can easily obtain the following:

Corollary 3.1. *Let* $f \in CON_\varepsilon[0,1]$ *and* $0 \leq m_1 < m_2 \leq 1$. *Let* P *be an* $\varepsilon-$*parabola connecting* $(m_1, f(m_1))$ *and* $(m_2, f(m_2))$. *Define*

$$g(m) := \begin{cases} P(m) & \text{if } m_1 \leq m \leq m_2 \\ f(m) & \text{otherwise} \end{cases} \ .$$

Then $g \in CON_\varepsilon[0,1]$ *as well.*

Corollary 3.2. *If* f *is sequentially* C^2, *convex function which satisfies* $f'' \geq \varepsilon$ *at all but a countable number of points, then* f *is* $\varepsilon-$*convex.*

Definition 3.2. *Let* Ψ *be a continuous function on* $[0,1]$.

- *A point* $m \in [0,1]$ *is called an* $\Psi-$*cluster point if there exists* $m_1 < m < m_2$ *so that*

$$s\Psi(m_1) + (1-s)\Psi(m_2) \leq \Psi\left(sm_1 + (1-s)m_2\right) - \varepsilon s(1-s)/2 \tag{3.1}$$

  holds for any $s \in [0,1]$.

- *A point* $m \in [0,1]$ *is called an* $\Psi-$*exposed point if for any* $m_1 < m < m_2$

$$\Psi(m) < \frac{m_2 - m}{m_2 - m_1}\Psi(m_1) + \frac{m - m_1}{m_2 - m_1}\Psi(m_2) - \varepsilon\frac{(m_2 - m)(m - m_1)}{(m_2 - m_1)^2} \ . \tag{3.2}$$

- *The set of all cluster points of* Ψ *is denoted by* C_Ψ. *The set of all exposed points of* Ψ *is denoted by* E_Ψ. *It's closure in* $[0,1]$ *is* $\overline{E}_\Psi$.

Remark 3.2. *i) Condition*(3.1) *states that the graph of* Ψ *at* (m_1, m_2) *is below the* $\varepsilon-$*parabola connecting* $(m_1, \Psi(m_1))$ *and* $(m_2, \Psi(m_2))$.

ii) Condition (3.2) *can be stated as follows: The point* $(m, \Psi(m))$ *is below the* $\varepsilon-$*parabola connecting the points* $(m_1, \Psi(m_1))$ *and* $(m_2, \Psi(m_2))$.

By Lemma 3.1-(ii) we obtain

Corollary 3.3. *If $\Psi \in CON_\varepsilon[0,1]$ then the equality holds in (3.1). In particular, Ψ coincides with an $\varepsilon-$parabola on any interval contained in C_Ψ.*

Lemma 3.2. *1. The set of cluster points C_Ψ is open.*

2. *$[0,1] = C_\Psi \cup \overline{E}_\Psi$ and $C_\Psi \cap \overline{E}_\Psi = \emptyset$.*
3. *If $\Psi_\varepsilon < \Psi$ on some interval (m_1, m_2), then Ψ_ε coincides with an $\varepsilon-$parabola on (m_1, m_2).*
4. *If $m \in \overline{E}_{\Psi_\varepsilon}$ then $\Psi_\varepsilon(m) = \Psi(m)$.*
5. *$C_\Psi = C_{\Psi_\varepsilon}$. In particular, $\overline{E}_{\Psi_\varepsilon} = \overline{E}_\Psi$.*
6. *If $m \in \overline{E}_\Psi$ then $\Psi_\varepsilon(m) = \Psi(m)$. Moreover, the function Ψ_ε is determined everywhere by Ψ on E_Ψ.*
7. *If $(m_1, m_2) \subset C_\Psi$ is a maximal interval[1] of C_Ψ, then (3.1) holds for any $m \in (m_1, m_2)$.*

Lemma 3.3. *$m \in E_\Psi$ if and only if (3.2) holds for any $m_1 < m < m_2$ which satisfy $m_i \in \overline{E}_\Psi$, $i = 1, 2$.*

Definition 3.3. *If V is a continuous function and Ψ is absolutely continuous on $[0,1]$, then V_Ψ is a continuous function defined by:*
$V_\Psi(m) := V(m)$ if $m \in \overline{E}_\Psi$, V_Ψ is a linear function on any interval $(m_1, m_2) \subset C_\Psi$.

We now define the *propagator* of a $\varepsilon-$convex function Ψ on $[0,1]$, *given V*:

Definition 3.4. *If Ψ is $\varepsilon-$convex and V absolutely continuous function on $[0,1]$, then $\mathcal{F}^V_{(t)}[\Psi] := [\Psi + tV_\Psi]_\varepsilon$.*

Lemma 3.4. *If $t > \tau$ then $C_{\mathcal{F}^V_t[\Psi]} \supseteq C_{\mathcal{F}^V_\tau[\Psi]}$. In particular, $\overline{E}_{\mathcal{F}^V_t[\Psi]} \subseteq \overline{E}_{\mathcal{F}^V_\tau[\Psi]}$ by Lemma 3.2-(2).*

We now claim the semigroup property of $\mathcal{F}^V$:

Proposition 3.1. *Given a continuous V and $\varepsilon-$convex function Ψ on $[0,1]$, for any $t \geq \tau \geq 0$*

$$\mathcal{F}^V_{(t-\tau)}\left[\mathcal{F}^V_{(\tau)}[\Psi]\right] = \mathcal{F}^V_{(t)}[\Psi] \ . \tag{3.3}$$

Finally, we shall need the following:

Lemma 3.5. *If $\{\Psi_N\}$ is a sequences of continuous function which converges uniformly to Ψ on $[0,1]$, then $\{[\Psi_N]_\varepsilon\}$ converges uniformly to $[\Psi]_\varepsilon$.*

[1] That is, if $J \supseteq (m_1, m_2)$ is an interval contained in C_Ψ, then $J = (m_1, m_2)$.

4 Proof of the Main Theorem

Set

$$\overline{\rho}_N := \rho_N(,0) \quad , \quad \overline{u}_N := u_N(,0) \ , \quad M_N(x,t) := \int_{-\infty}^x \rho_N(s,t)ds$$

and $X_N(m,t)$ the generalized inverse of M_N as a function of x. Let

$$\widetilde{\Psi}^{(N)}(m,t) := \int_0^m X_N(s,t)ds \quad , \quad \overline{\Psi}^{(N)}(m) := \widetilde{\Psi}^{(N)}(m,0)$$

$$\widetilde{\Phi}^{(N)}(x,t) = \left(\widetilde{\Psi}^{(N)}\right)^*(x,t) = \int_{-\infty}^x M_N(s,t)ds \ , \quad \overline{\Phi}^{(N)}(x) := \widetilde{\Phi}^{(N)}(x,0) \ . \tag{4.1}$$

$$v_N(m) = \overline{u}_N\left(\partial_m \overline{\Psi}^{(N)}(m)\right) \ ; \quad \overline{V}^{(N)}(m) = \int_0^m v_N(s)ds \tag{4.2}$$

and

$$\Psi^{(N)}(m,t) := \mathcal{F}^{(N)}_{(t)}\left[\overline{\Psi}^{(N)}\right](m) \ ; \quad \Phi^{(N)}(x,t) = \left(\Psi^{(N)}\right)^*(x,t) \ ,$$

where

$$\mathcal{F}^{(N)}_{(t)} := \mathcal{F}^{\overline{V}^{(N)}}_{(t)} \ .$$

By (4.1), (4.2) and the law of collision (1.7) we obtain

$$\partial_x^2 \widetilde{\Phi}^{(N)} = \partial_x M_N = \rho_N \quad ; \quad v_N\left(\partial_x \widetilde{\Phi}^{(N)}\right) = u_N \ . \tag{4.3}$$

Our object is to show

$$\widetilde{\Psi}^{(N)} = \Psi^{(N)} \ \text{ i.e } \ \widetilde{\Phi}^{(N)} = \Phi^{(N)} \tag{4.4}$$

for any N, and that

$$\lim_{N\to\infty} \widetilde{\Phi}^{(N)}(,t) = \Phi(,t) \quad , \quad res. \quad \lim_{N\to\infty} \widetilde{\Psi}^{(N)}(,t) = \Psi(,t) \tag{4.5}$$

exists locally uniformly on $\mathbb{R}$ (res. uniformly on $[0,1]$) for any $t > 0$.

Granted (4.4) and (4.5), we can prove the Theorem as follows:
By assumption (2.1) and the law of collisions (1.7), the supports of $\rho_N(,t)$ are all contained in a compact set $K_t \subset \mathbb{R}$. The limit Φ in (4.5) is clearly a convex function and defines a density $\rho = \partial_x^2\Phi$ of a probability measure for any $t \geq 0$, which is also supported in K_t. By (4.3) and (4.5) it follows that ρ is the weak limit of ρ_N for any $t \geq 0$.

Now, the limit

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \overline{u}_N(x)\overline{\rho}_N(x)\phi(x)dx = \int_{-\infty}^{\infty} \overline{u}(x)\overline{\rho}(x)\phi(x) \tag{4.6}$$

holds by assumption. We change the variable x into $m = \partial_x\overline{\Phi}^{(N)}$. Recall that $x = \partial_m\overline{\Psi}^{(N)}$ is the inverse relation and using $\overline{\rho}_N = \partial_x^2\overline{\Phi}^{(N)}$, $\overline{\rho} = \partial_x^2\overline{\Phi}$ and (4.2) we rewrite (4.6) as

$$\lim_{N\to\infty} \int_{-1}^1 v_N(m)\phi\left(\partial_m\overline{\Psi}^{(N)}\right)dm = \int_{-1}^1 v(m)\phi\left(\partial_m\overline{\Psi}\right)dm \ . \tag{4.7}$$

From (4.5) evaluated at $t=0$ we obtain $\overline{\Phi}^{(N)} \to \overline{\Phi}$ locally uniformly on $\mathbb{R}$. Since $\overline{\Psi}^{(N)}$ (res. $\overline{\Psi}$) are the Legendre transforms of $\overline{\Phi}^{(N)}$ (res. $\overline{\Phi}$), it also follows that $\overline{\Psi}^{(N)} \to \overline{\Psi}$ uniformly in $[0,1]$. Moreover, since $\overline{\Psi}^{(N)}$ are convex it follows that $\partial_m \overline{\Psi}^{(N)} \to \partial_m \overline{\Psi}$ strongly in $L^1[0,1]$. Recall that the sequence v_N is uniformly bounded in $L^\infty[0,1]$ by assumption. From this, (4.7) and the obtained L^1 convergence $\partial_m \overline{\Psi}^{(N)} \to \partial_m \overline{\Psi}$ we obtain that v is the *unique* weak L^∞ limit of v_N.

We have to prove the existence of $u = u(x,t)$ for which

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} u_N(x,t)\rho_N(x,t)\phi(x)dx = \int_{-\infty}^{\infty} u(x,t)\rho(x,t)\phi(x) \tag{4.8}$$

holds for all $t>0$ and $\phi \in C_0(\mathbb{R})$. We note that by (4.1) and (4.2) and

$$u_N(x,t) = v_N\left(\partial_x \widetilde{\Phi}^{(N)}(x,t)\right) . \tag{4.9}$$

Using the change the variable x into $m = \partial_x \widetilde{\Phi}^{(N)}$, recalling that $x = \partial_m \widetilde{\Psi}^{(N)}$ is the inverse relation and using (4.3), (4.1) and (4.9) we write the left side of (4.8) as

$$\lim_{N\to\infty} \int_{-1}^{1} v_N(m)\phi\left(\partial_m \widetilde{\Psi}^{(N)}(m,t)\right) dm \tag{4.10}$$

By the same argument as above we observe, using (4.5), that $\partial_m \widetilde{\Psi}^{(N)}(,t) \to \partial_m \Psi(,t)$ in $L^1[0,1]$ for any fixed $t>0$. Since ϕ is continuous it follows that (4.10) equals

$$\lim_{N\to\infty} \int_{-1}^{1} v_N(m)\phi\left(\partial_m \Psi(m,t)\right) dm \ . \tag{4.11}$$

Since we know already that the weak limit in $L^\infty[0,1]$ of v_N is v, it follows that (4.11) equals

$$\int_{-1}^{1} v(m)\phi\left(\partial_m \Psi(m,t)\right) dm \ ,$$

which implies that (4.6) is satisfied were

$$u(x,t) = v\left(\partial_x \Phi(x,t)\right) \ .$$

This verifies the second claim in (2.3).

We now turn to the proofs of (4.4) and (4.5):
Let $\{t_l\}$ be the set of collision times corresponding to (ρ_N, u_N). This implies that there exists sets $J_l \subset \{1, \dots N-1\}$ where

i) $x_{i+1}(t) - x_i(t) > \nu_N$ for any $0 \le t < t_l$ and $i \in J_l$.

ii) $x_{i+1}(t_l) - x_i(t_l) = \nu_N$ for any $i \in J_l$.

iii) If $i \notin J_l$ then either $x_{i+1}(t) - x_i(t) > \nu_N$ or $x_{i+1}(t) - x_i(t) = \nu_N$ for any $t_{l-1} \le t \le t_l$, were $t_0 \equiv 0$.

Now, we observe that for any $t \in [t_l, t_{l+1}]$,

$$\widetilde{\Psi}^{(N)}(,t) = \widetilde{\Psi}^{(N)}(,t_l) + (t-t_l)V_{\widetilde{\Psi}^{(N)}(,t_l)} \tag{4.12}$$

is $\varepsilon-$convex. To see this, note that $\partial_m \widetilde{\Psi}^{(N)}(,t) \equiv X_N(,t)$ is monotone non-decreasing in m for $t \in [t_l, t_{l+1}]$. Indeed, $X_N(,t)$ is the generalized inverse of $M_N(,t)$ which is monotone by definition. In addition, any $m \in [0,1]$ which is not an integer multiple of ν_N is contained in $C_{\widetilde{\Psi}^{(N)}(,t_l)}$, so, by Corollary 3.3, $\partial^2_m \widetilde{\Psi}^{(N)}(m,t) = \varepsilon$ for all but a finite number of m. It follows that $\widetilde{\Psi}^{(N)}(,t)$ is $\varepsilon-$convex by Corollary 3.2.

We now proceed to the proof of (4.4) by induction. At $t_0 = 0$ we get

$$\widetilde{\Psi}^{(N)}(m,t_0) = \Psi^{(N)}(m,t_0) = \overline{\Psi}^{(N)}(m)$$

by definition. From the $\varepsilon-$convexity of (4.12) we obtain

$$\Psi^{(N)}(m,t) = \left[\widetilde{\Psi}^{(N)}(,0) + tV_{\widetilde{\Psi}^{(N)}(,0)}\right]_\varepsilon (m) = \widetilde{\Psi}^{(N)}(m,t)$$

for $t_0 \le t \le t_1$. Suppose now that we verified $\Psi^{(N)}(m,t) = \widetilde{\Psi}^{(N)}(m,t)$ for $t \le t_j$. Then (4.12) implies

$$\widetilde{\Psi}^{(N)}(,t) = \left[\widetilde{\Psi}^{(N)}(,t_j) + (t-t_j)V_{\widetilde{\Psi}^{(N)}(,t_j)}\right]_\varepsilon := \mathcal{F}^{(N)}_{(t-t_j)}\left[\widetilde{\Psi}^{(N)}(,t_j)\right](m)$$

for $t_j \le t \le t_{j+1}$. But $\widetilde{\Psi}^{(N)}(,t_j) = \Psi^{(N)}(,t_j) \equiv \mathcal{F}^{(N)}_{(t_j)}\left[\overline{\Psi}^{(N)}\right]$ by the induction hypothesis, so

$$\widetilde{\Psi}^{(N)}(m,t) = \mathcal{F}^{(N)}_{(t-t_j)}\left[\mathcal{F}^{(N)}_{(t_j)}\left[\overline{\Psi}^{(N)}\right]\right](m) \ .$$

This verifies

$$\widetilde{\Psi}^{(N)}(,t) = \mathcal{F}^{(N)}_{(t)}\left[\overline{\Psi}^{(N)}\right] \equiv \Psi^{(N)}(,t)$$

for $t_j \le t \le t_{j+1}$ by Proposition 3.1.

We now prove (4.5): From the weak convergence $\overline{\rho}_N \to \overline{\rho}$ we obtain the $L^1[0,1]$ convergence $\overline{X}_N := X_N(,0) \to \overline{X}$. Since $\overline{X}_N \equiv \partial_m \overline{\Psi}^{(N)}$ it follows that

$$\lim_{N\to\infty} \overline{\Psi}^{(N)} = \overline{\Psi} \tag{4.13}$$

uniformly on $[0,1]$ as well.

Next, we already proved that v_N converges weakly in $L^\infty[0,1]$ to some $v \in L^\infty[0,1]$. This and (4.2) imply that

$$\lim_{N\to\infty} \overline{V}^{(N)} = \overline{V} \tag{4.14}$$

uniformly on $[0,1]$.

Next, $\partial^2_x \overline{\Phi} = \overline{\rho}(x) < \varepsilon^{-1}$ by assumption, so $\partial^2_m \overline{\Psi} > \varepsilon$ by duality. This implies that $C_{\overline{\Psi}} = \emptyset$, so $\overline{V} = \overline{V}_{\overline{\Psi}}$.

Next, we claim

$$\lim_{N\to\infty} \left\| V^{(N)}_{\overline{\Psi}^{(N)}} - V^{(N)} \right\|_\infty = 0 \ , \tag{4.15}$$

which, together with (4.14), implies the uniform convergence

$$\lim_{N\to\infty} V^{(N)}_{\overline{\Psi}^{(N)}} = V_{\overline{\Psi}} \ . \tag{4.16}$$

From (4.13) and (4.16) we obtain the uniform convergence of

$$\lim_{N\to\infty} \left(\overline{\Psi}^{(N)} + t\overline{V}^{(N)}_{\overline{\Psi}^{(N)}} \right) = \overline{\Psi} + t\overline{V}_{\overline{\Psi}}$$

By Lemma 3.5 it follows that

$$\Psi^{(N)}(m,t) \equiv \left[\overline{\Psi}^{(N)} + t\overline{V}^{(N)}_{\overline{\Psi}^{(N)}} \right]_\varepsilon (m) \to \left[\overline{\Psi} + t\overline{V}_{\overline{\Psi}} \right]_\varepsilon (m) \equiv \Psi(m,t) \ ,$$

uniformly on $[0,1]$ as well. By (4.4) we obtain that

$$\lim_{N\to\infty} \widetilde{\Psi}^{(N)}(,t) = \Psi(,t)$$

uniformly on $[0,1]$. This, in turn, implies (4.5) by taking the Legendre transform of this sequence.

Finally, the claim (4.15) is verified as follows: Let $m \in [0,1]$. If $m \in E_{\overline{\Psi}^{(N)}}$ then $V^{(N)}_{\overline{\Psi}^{(N)}} = V_N$. If $m \notin E_{\overline{\Psi}^{(N)}}$ then $m \in Supp(\overline{\rho}_N)$. But, if (m_1, m_2) is an interval containing m and contained in $Supp(\overline{\rho}_N)$, then (m_1, m_2) must contain points *not in* the support of $\overline{\rho}_N$ for sufficiently large N, for, otherwise, the weak limit $\overline{\rho} = \varepsilon^{-1}$ on this interval, contradiction to the assumption $\|\overline{\rho}\|_\infty < \varepsilon^{-1}$. In particular, it follows that for sufficiently large N, any such interval must contain points of $E_{\overline{\Psi}^{(N)}}$, hence points for which $V^{(N)} = V^{(N)}_{\overline{\Psi}^{(N)}}$. The sequence $V^{(N)}$ is equi-continuous (since $\partial_m V^{(N)} = v_N$ and $\|v_N\|_\infty = \|\overline{u}_N\|_\infty < C$ by assumption). This verifies (4.15).

5 Proofs of auxiliary results

The proofs of Lemma 3.1 and Corollaries 3.1, 3.2 and 3.3 are rather easy and we skip it.

Proof. (of Lemma 3.2):
Part (1) is evident from definition.
(2) Let $m \notin C_\Psi$. Let $m_\alpha < m < m_\beta$. By (3.1) there exists a point $s_0 \in [0,1]$ for which

$$s_0\Psi(m_\alpha) + (1-s_0)\Psi(m_\beta) + \varepsilon s_0(1-s_0)/2 > \Psi\left(s_0 m_\alpha + (1-s_0)m_\beta\right) \ .$$

Since Ψ is continuous, the inequality above holds for some interval $(s_1, s_2) \subset [0,1]$ where $s_0 \in (s_1, s_2)$. Let $m_* = s_0 m_\alpha + (1-s_0)m_\beta$ and $m_i = s_i m_\alpha + (1-s_i)m_\beta$, $i = 1,2$. Then m_* satisfies

$$\Psi(m_*) < \frac{m_2 - m}{m_2 - m_1}\Psi(m_1) + \frac{m_* - m_1}{m_2 - m_1}\Psi(m_2) - \varepsilon\frac{(m_2 - m_*)(m_* - m_1)}{(m_2 - m_1)^2} \ .$$

By (3.2), $m_* \in E_\Psi$. On the other hand, $m_* \in (m_\alpha, m_2)$ which is an arbitrary neighborhood of m. Hence $m \in \overline{E}_\Psi$ and (2) follows.
(3) For any $m \in (m_1, m_2)$ let us consider a small interval $(m_\alpha, m_\beta) \subset (m_1, m_2)$ containing m. Let P the $\varepsilon-$parabola crossing the points $(m_\alpha, \Psi_\varepsilon(m_\alpha))$ and $(m_\beta, \Psi_\varepsilon(m_\beta))$. Now consider

$$Y(s) := \begin{cases} P(s) & s \in (m_\alpha, m_\beta) \\ \Psi_\varepsilon(s) & \text{otherwise} \end{cases}$$

Then Y is $\varepsilon-$convex by Corollary 3.1. We may choose the interval (m_α, m_β) so small, for which $Y < \Psi$ on this interval. In particular Y is an $\varepsilon-$convex function which satisfies $Y \le \Psi$ on [0.1]. Hence $\Psi_\varepsilon \ge Y$ on $[0,1]$ by Definition 2.2. On the other hand, since Ψ_ε is $\varepsilon-$convex then $\Psi_\varepsilon \le P$ on (m_α, m_β).It follows that $\Psi_\varepsilon = P$ on (m_α, m_β). The proof follows since the same argument can be applied for any $m \in (m_1, m_2)$.
(4)- Let $m \in \overline{E}_{\Psi_\varepsilon}$. Note that $\Psi_\varepsilon(m) \le \Psi(m)$. Suppose $\Psi_\varepsilon(m) < \Psi(m)$. Let (m_1, m_2) be the maximal interval containing m on which $\Psi_\varepsilon < \Psi$. In particular, $\Psi(m_i) = \Psi_\varepsilon(m_i)$ for $i = 1, 2$. By (3), Ψ_ε coincides with an $\varepsilon-$parabola on the interval (m_1, m_2). This implies that Ψ_ε satisfies condition (3.1) at m, contradicting $m \in \overline{E}_{\Psi_\varepsilon}$ via point 2.
(5) - Let $m \in C_\Psi$, and m_1, m_2 as in (3.1). Let P the $\varepsilon-$parabola crossing the points $(m_1, \Psi(m_1))$ and $(m_2, \Psi(m_2))$. Now consider

$$Y(s) := \begin{cases} P(s) & s \in (m_1, m_2) \\ \Psi_\varepsilon(x) & \text{otherwise} \end{cases}$$

It follows by (***) that Y is an $\varepsilon-$convex function. Moreover, $Y \le \Psi$ since $P \le \Psi$ on (m_1, m_2) and $\Psi_\varepsilon \le \Psi$ everywhere. By Definition 2.2, $Y \le \Psi_\varepsilon$. In particular, $P \le \Psi_\varepsilon$ on the interval (m_1, m_2), which implies (3.1), so $C_\Psi \subseteq C_{\Psi_\varepsilon}$.
Conversely, let $m \in C_{\Psi_\varepsilon}$. Set (m_1, m_2) a maximal interval of C_{Ψ_ε}. Then $m_i \in \overline{E}_{\Psi_\varepsilon}$ for $= 1, 2$. By point (4) $\Psi_\varepsilon(m_i) = \Psi(m_i)$, $i = 1, 2$. Since $\Psi_\varepsilon \le \Psi$ on $[0,1]$ (in particular, on (m_1, m_2)), and (3.1) is satisfied (with an equality) on (m_1, m_2) for Ψ_ε by Corollary 3.3, it follows that (3.1) is also satisfied for Ψ on (m_1, m_2). In particular $m \in C_\Psi$, so $C_\Psi \subseteq \overline{C}_\Psi$.
6 - The first part follows from points 4 and 5. The second part from point 3.
7- Follows from Corollary 3.3. □

Proof. (of Lemma 3.3)
The "only if" part is trivial from definition. For the "if" part, let $m_1 < m < m_2$, and assume first that $m_2 \in \overline{E}_\Psi$ while $m_1 \in C_\Psi$.

Let Q be the $\varepsilon-$parabola connecting $(m_1, \Psi(m_1))$ to $(m_2, \Psi(m_2))$. We show that $Q(m) > \Psi(m)$. This is equivalent to (3.2) for m_1, m_2.

Let $m_\alpha < m_1 < m_\beta$ be a maximal interval of C_Ψ containing m_1. Since $m \in E_\Psi$ by assumption, then $m_\beta < m$. Also, $m_\alpha, m_\beta \in \overline{E}_\Psi$ so, by the assumption of the Lemma, (3.2) is satisfied where m_1 is replaced by m_α or m_β, respectively.

Now, let P_α be the $\varepsilon-$parabola connecting the points $(m_\alpha, \Psi(m_\alpha))$ and $(m_2, \Psi(m_2))$. Likewise, P_β is the $\varepsilon-$parabola connecting the points $(m_\beta, \Psi(m_\beta))$ and $(m_2, \Psi(m_2))$ and P the $\varepsilon-$parabola connecting the points $(m_1, \Psi(m_1))$ and $(m_2, \Psi(m_2))$. Since $m_2 \in \overline{E}_\Psi$ and both $m_\alpha, m_\beta \in \overline{E}_\Psi$, the condition of the Lemma holds for both intervals (m_α, m_2) and

(m_β, m_2). It then follows by the assumption of the Lemma that

$$\Psi(m) < \min\{P_\alpha(m), P_\beta(m)\} \ . \tag{5.1}$$

In addition, $\Psi(m_1) \geq P(m_1)$ since (m_α, m_β) is a maximal interval of C_Ψ and Lemma 3.2-(7) applies.
However, $P(m_1) \geq \min\{P_\alpha(m_1), P_\beta(m_1)\}$. Hence $Q(m_1) \geq \min\{P_\alpha(m_1), P_\beta(m_1)\}$. Recalling that any 2 $\varepsilon-$parabolas may intersect in, at most, one point, and that $Q(m_2) = P_\alpha(m_2) = P_\beta(m_2)$, it follows that $Q(s) \geq \min\{P_\alpha(s), P_\beta(s)\}$ for $m_2 \geq s \geq m_1$. In particular, $Q(m) > \Psi(m)$ by (5.1).

In a similar way we remove the condition $m_2 \in \overline{E}_\Psi$ and prove $Q(m) > \Psi(m)$ for any $m_1 < m < m_2$. This implies $m \in E_\Psi$ by definition. $\square$

Proof. (of Lemma 3.4)
Let $m \in C_{\mathcal{F}^V_\tau[\Psi]}$. By Lemma 3.2-(5) $m \in C_{\Psi+\tau V_\Psi}$. By Definition 3.2 there exists an interval (m_1, m_2) containing m where

$$\begin{aligned} s\left[\Psi(m_1) + \tau V_x(m_1)\right] + (1-s)\left[\Psi(m_2) + \tau V_\Psi(m_2)\right] \\ \leq \left[\Psi + \tau V_\Psi\right](sm_1 + (1-s)m_2) - \varepsilon s(1-s)/2 \ , \end{aligned} \tag{5.2}$$

for any $s \in [0, 1]$. That is,

$$\begin{aligned} \tau\left[sV_\Psi(m_1) + (1-s)V_\Psi(m_2) - V_\Psi\left(sm_1 + (1-s)m_2\right)\right] \\ \leq \Psi\left(sm_1 + (1-s)m_2\right) - s\Psi(m_1) - (1-s)\Psi(m_2) - \varepsilon s(1-s)/2 \ . \end{aligned} \tag{5.3}$$

Since Ψ is $\varepsilon-$convex, the RHS of (5.3) is non-positive. Hence, the LHS of (5.3) is non-positive as well. It then follows that if we replace τ by $t > \tau$ on the left of (5.3), the inequality will survive. This implies that (3.1) holds for m_1, m_2 where Ψ is replaces by $\Psi + tV_\Psi$. Then $m \in C_{\Psi+tV_\Psi}$ as well. The Lemma follows since $C_{\Psi+tV_\Psi} = C_{\mathcal{F}^V_{(t)}[\Psi]}$ by Lemma 3.2-(5) again. $\square$

Proof. (of Proposition 3.1)
Set $Y = \Psi + \tau V_\Psi$, $Z = \Psi + tV_\Psi$ and $W = Y_\varepsilon + (t-\tau)V_{Y_\varepsilon}$. We shall prove that

$$C_W = C_Z \ . \tag{5.4}$$

Granted (5.4) we obtain $\overline{E}_Z = \overline{E}_W$ by Lemma 3.2-(2). Recall that, by Lemma 3.2-(4, 5), if $m \in \overline{E}_Z$ then $Z(m) = Z_\varepsilon(m) = \mathcal{F}^V_{(t)}[\Psi](m)$. Also, $m \in \overline{E}_Y$ by Lemma 3.4, so $Y(m) = Y_\varepsilon(m)$, $V_\Psi(m) = V_{Y_\varepsilon}(m)$ hence $W(m) = Y(m) + (t-\tau)V_\Psi(m) = Z(m)$. Hence, by (5.4) we obtain $Z_\varepsilon(m) = W_\varepsilon(m)$ for any $m \in \overline{E}_Z = \overline{E}_W$. This implies $Z_\varepsilon = W_\varepsilon$ everywhere by Lemma 3.1-(6), which implies (3.3).
Proof of (5.4):
Let $m \in C_Z$. Let $(m_1, m_2) \subset C_\Psi$ be a maximal interval of C_Ψ. By Lemma 3.2-(7)

$$\begin{aligned} t\left[V_\Psi(sm_1 + (1-s)m_2) - sV_\Psi(m_1) - (1-s)V_\Psi(m_2)\right] \\ \leq \left[s\Psi(m_1) + (1-s)\Psi(m_2)\right] - \Psi(sm_1 - (1-s)m_2) - \varepsilon s(1-s)/2 \end{aligned} \tag{5.5}$$

holds for any $s \in [0,1]$. In turn, (5.5) implies

$$(t-\tau)\left[V_\Psi(sm_1+(1-s)m_2)-sV_\Psi(m_1)-(1-s)V_\Psi(m_2)\right] \\ \leq [sY(m_1)+(1-s)Y(m_2)]-Y(sm_1+(1-s)m_2)-\varepsilon s(1-s)/2 \quad (5.6)$$

for any $s \in [0,1]$. Since $\overline{E}_Y \subseteq \overline{E}_\Psi$ by Lemma 3.4 we obtain that $V_Y(sm_1+(1-s)m_2) = V_\Psi(sm_1+(1-s)m_2)$ whenever $sm_1+(1-s)m_2 \in \overline{E}_Y$. Moreover, $Y_\varepsilon(sm_1+(1-s)m_2,\tau) = Y(sm_1+(1-s)m_2)$ under the same condition. Since (m_1,m_2) is assumed a *maximal* interval of C_Z, it follows by Lemma 3.2-(2) that $m_1,m_2 \in \overline{E}_Z$. However, $\overline{E}_Z \subseteq \overline{E}_Y$ (Lemma 3.4 again), so $m_1,m_2 \in \overline{E}_Y$ and $V_Y(m_i)=V_\Psi(m_i)$, $Y_\varepsilon(m_i)=Y(m_i)$ for $i=1,2$ as well. It then follows from (5.6) that

$$(t-\tau)\left[V_Y(sm_1+(1-s)m_2,\tau)-sV_Y(m_1,\tau)-(1-s)V_Y(m_2,\tau)\right] \\ -[sY_\varepsilon(m_1,\tau)+(1-s)Y_\varepsilon(m_2,\tau)]+Y_\varepsilon(sm_1-(1-s)m_2,\tau)+\varepsilon s(1-s)/2 \geq 0 \quad (5.7)$$

holds for any such s. But, on the complement of $\overline{E}_Y$ in (m_1,m_2), the RHS of (5.7) is linear in s. Hence, the inequality (5.7) holds for any $s \in [0,1]$. This, in turn, implies that (3.1) is satisfied where Ψ replaced by W. Thus, $(m_1,m_2) \subset C_W$ so $C_Z \subseteq C_W$. In particular $\overline{E}_W \subseteq \overline{E}_Z$.

If $m \in \overline{E}_Z$, then $Z(m) = \Psi(m)+tV_\Psi(m) = Y(m)+(t-\tau)V_\Psi(m)$. On the other hand, Lemma 3.4 we know that $\overline{E}_Z \subset \overline{E}_Y$, so $m \in \overline{E}_Y$ and by Definition 3.3 we obtain $V_\Psi(m)=V_Y(m)$. Lemma 3.2-(6) also yields $Y_\varepsilon(m)=Y(m)$. Hence

$$Z(m) = Y_\varepsilon(m)+(t-\tau)V_Y(m) = W(m)\ . \quad (5.8)$$

Suppose now $m \in E_Z$. Hence, by (5.8) and Definition 3.2,

$$W(m) = Z(m) < \frac{m_2-m}{m_2-m_1}Z(m_1)+\frac{m-m_1}{m_2-m_1}Z(m_2)-\varepsilon\frac{(m_2-m)(m-m_1)}{2(m_2-m_1)^2}\ . \quad (5.9)$$

for *any* $m_1 < m < m_2$. Suppose, in addition, $m_i \in \overline{E}_W$, $i=1,2$. Since we know $\overline{E}_W \subseteq \overline{E}_Z$ then $m_1,m_2 \in \overline{E}_Z$. In particular, $W(m_i)=Z(m_i)$ by (5.8), so (5.9) implies

$$W(m) < \frac{m_2-m}{m_2-m_1}W(m_1)+\frac{m-m_1}{m_2-m_1}W(m_2)-\varepsilon\frac{(m_2-m)(m-m_1)}{2(m_2-m_1)^2}\ . \quad (5.10)$$

Since (5.10) holds for any $m_i \in \overline{E}_W$, it implies that $z \in E_W$ by Lemma 3.3. This implies $E_Z \subseteq E_W$ and complete the proof of (5.4). □

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